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Parametrically excited, standing cross-waves

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Luke's (1967) variational formulation for surface waves is extended to incorporate the motion of a wavemaker and applied to the cross-wave problem. Whitham's average-Lagrangian method then is invoked to obtain the evolution equations for the slowly varying complex amplitude of the parametrically excited cross-wave that is associated with symmetric excitation of standing waves in a rectangular tank of width π/k, length l and depth d for which $kl = O(1)$ and $kd \gg 1$. These evolution equations are Hamiltonian and isomorphic to those for parametric excitation of surface waves in a cylinder that is subjected to a vertical oscillation, for which phase-plane trajectories, stability criteria and the effects of damping are known (Miles 1984*a*). The formulation and results differ from those of Garrett (1970) in consequence of his linearization of the boundary condition at the wavemaker and his neglect of self-interaction of the cross-waves in the free-surface conditions (although Garrett does incorporate self-interaction in his calculation of the equilibrium amplitude of the cross-waves). These differences have only a small effect on the criterion for the stability of plane waves, but the self-interaction is crucial for the determination of the stability of the cross-waves.

1. Introduction

Cross-waves are induced by a symmetric wavemaker in a rectangular channel when the frequency of excitation approximates twice one of the resonant frequencies of the transverse standing-wave modes and the amplitude of excitation exceeds a certain threshold. The problem is one of parametric resonance, in which energy is transferred from the symmetric (with respect to the vertical mid-plane of the channel) motion to the antisymmetric cross-waves through nonlinear interactions. It has been studied by Garrett (1970) on the assumption of standing waves in a short tank (length $\lesssim$ breadth) and by Mahony (1972) and Jones (1984) on the assumption of progressive waves in a long tank (length $\gg$ breadth). Both Garrett and Mahony linearize the boundary condition at the wavemaker and neglect the self-interaction of the cross-waves in the free-surface conditions (although Garrett does incorporate self-interaction in his calculation of the equilibrium amplitude of the cross-waves). The former approximation appears to have only a small effect, and the neglect of self-interaction no effect, on the criterion for the stability of plane waves (cf. Jones), but self-interaction is crucial for the determination of the stability of the cross-waves.

The somewhat simpler problem of parametric excitation of cross-waves through vertical oscillation of a channel has been studied by Larraza & Putterman (1984) and Miles (1984*b*). I used Whitham's (1974) average-Lagrangian method and obtained results in quantitative agreement with the observations of Wu, Keolian & Rudnick (1984), although only after incorporating capillarity and dissipation. Whitham's method avoids many of the complications of a secular perturbation solution, such as those of Larraza & Putterman and Jones. I present here the application of that method to the cross-wave problem.

Following Garrett (but with independent notation), I consider the excitation of gravity waves of free-surface displacement ζ in a rectangular tank of width b, depth d and length l in response to the wavemaker motion

$$x = \chi(z, t) = af(z)\sin 2\omega t \quad (0 < y < b, \quad -d < z < \zeta) \tag{1.1}$$

on the assumptions that

$$ka \equiv \epsilon \ll 1, \quad kd \gg 1, \quad kl = O(1) \quad (k \equiv \pi/b). \tag{1.2}$$

It is evident from symmetry that the boundary-value problem admits a plane-wave (y-independent) solution; however, nonlinearity may couple energy into cross-waves if ω approximates one of the natural frequencies

$$\omega_n = (ngk)^{\frac{1}{2}} \quad (n = 1, 2, \ldots, \quad kd \gg 1). \tag{1.3}$$

I assume that ω approximates ω_1 according to

$$\frac{\omega^2-\omega_1^2}{\omega^2} = 1-\frac{k}{\kappa} = O(\epsilon) \quad \left(\kappa \equiv \frac{\omega^2}{g}\right), \tag{1.4}$$

which determines the bandwidth of the hypothetical resonance. If $\omega \approx \omega_n$ ($n = 2, 3, \ldots$) it is necessary only to replace k by nk in (1.4) and subsequently. The dominant effect of a small surface tension T is to raise the natural frequency, with the result that ω_1^2 and k in (1.4) are multiplied by $1+\hat{T}$, where

$$\hat{T} \equiv \frac{k^2T}{\rho g} \ll 1. \tag{1.5}$$

This correction may imply $O(1)$ effects for $\epsilon \ll 1$, whereas the remaining effects of small surface tension are uniformly $O(\hat{T})$ relative to unity.

The free-surface displacement of the hypothetical cross-wave, which is superimposed on the plane wave, may be posed in the form

$$\zeta = \epsilon^m a \mathscr{R}\{(p+\mathrm{i}q)\,\mathrm{e}^{-\mathrm{i}\omega t}\}\sqrt{2}\,\cos ky + o(\epsilon^m a), \tag{1.6}$$

where $p+\mathrm{i}q$ is a dimensionless, slowly varying complex amplitude. The principal aim of the analysis is to determine the evolution equations for (p, q), which, in turn, determine the stability, or otherwise, of the underlying plane-wave motion. If $kl = O(1)$, as in the present analysis, $m = -\frac{1}{2}$, so that the cross-wave dominates the plane wave, and the slow timescale is $1/\epsilon\omega$. If $kl = O(1/\epsilon)$, as is implicit in the analyses of Mahony (1972) and Jones (1984), $m = 0$, so that the cross-wave and the plane wave have similar magnitudes, the slow timescale is $1/\epsilon^2\omega$, and $p+\mathrm{i}q$ also exhibits a spatial variation with the lengthscale $1/\epsilon k$.

I begin my analysis, in §2, by extending Luke's (1967) variational formulation for surface waves to the wavemaker problem (or to other problems with moving boundaries) and transforming his Lagrangian to the sum of integrals over the free surface and the wavemaker plus a volume integral that vanishes if the trial function for the velocity potential ϕ satisfies Laplace's equation. In §3, I derive appropriate trial functions for ϕ and the free-surface displacement ζ by combining Havelock's (1929) solution of the basic wavemaker problem with Rayleigh's (1915) solution of the nonlinear standing-wave problem. I then calculate the average Lagrangian in §4 and the corresponding evolution equations in §5. The resulting system is Hamiltonian and, through a canonical transformation, isomorphic to the Hamiltonian system for parametric excitation of surface waves in a cylinder that is subjected to a vertical oscillation (Miles 1984*a*). My stability criterion for the plane-wave motion differs from Garrett's (1970) in consequence of his linearization of the boundary condition at the wavemaker; moreover, I obtain quantitative stability criteria for the cross-waves and phase-plane (p, q) trajectories and incorporate weak damping.

The only experimental results for cross-waves in a short tank of which I am aware are those of Lin & Howard (1960).† Both Garrett's and the present analytical results predict cross-wave-equilibrium amplitudes (or, equivalently, resonant frequency shifts) *vs.* wavemaker amplitude (see Garrett's figure 2) in qualitative agreement with Lin & Howard's measurements.

2. Variational formulation

The assumption of motion started from rest in an incompressible, inviscid fluid in the wave tank described in §1 leads to the boundary-value problem

$$\nabla^2\phi = 0 \quad (\chi < x < l, \quad 0 < y < b, \quad -d < z < \zeta), \tag{2.1}$$

$$\phi_z = \zeta_t + \nabla\phi\cdot\nabla\zeta, \quad \phi_t + \tfrac{1}{2}(\nabla\phi)^2 + g\zeta = 0 \quad (z = \zeta), \tag{2.2a, b}$$

$$\phi_x = 0 \quad (x = l), \quad \phi_y = 0 \quad (y = 0, b), \quad \phi_z = 0 \quad (z = -d), \tag{2.3a, b, c}$$

$$\phi_x = \chi_t + \nabla\phi\cdot\nabla\chi \quad (x = \chi) \tag{2.4}$$

for the velocity potential $\phi(x, y, z, t)$ and the free-surface displacement $\zeta(x, y, t)$, where the subscripts x, y, z, t signify partial differentiation. The boundary condition (2.3*c*) is imposed at $z = \infty$ (deep-water waves)in §§3–5.

The boundary-value problem (2.1)–(2.4) may be deduced from the variational principle

$$\delta J = 0, \quad J \equiv \int \hat{L}\,\mathrm{d}t, \tag{2.5a, b}$$

where J is the action integral of the Lagrangian (Luke 1967)

$$\hat{L} = -\iiint [\phi_t + \tfrac{1}{2}(\nabla\phi)^2 + gz]\,\mathrm{d}V, \tag{2.6}$$

and the volume integral is over the domain bounded by the wavemaker $(x = \chi)$, the free surface $(z = \zeta)$ and the fixed boundaries $(x = l, y = 0, b$ and $z = -d)$. The proof follows Luke (1967) and Whitham (1974) after allowing for the motion of the wavemaker. Invoking the identities (the second of which is Green's theorem)

$$\iiint \delta\phi_t\,\mathrm{d}V = \partial_t \iiint \delta\phi\,\mathrm{d}V + \iint n_t\,\delta\phi\,\mathrm{d}S, \tag{2.7}$$

$$\iiint \nabla\phi\cdot\delta\nabla\phi\,\mathrm{d}V = -\iiint (\nabla^2\phi)\,\delta\phi\,\mathrm{d}V - \iint \phi_n\,\delta\phi\,\mathrm{d}S, \tag{2.8}$$

† Barnard & Pritchard (1972) have carried out extensive experiments for cross-waves in a long $(kl = O(1/\epsilon))$ tank.

where S is the bounding surface, n is the inwardly directed normal to S, and ϕ_n is the normal derivative, we obtain

$$\delta\hat{L}+\partial_t\iiint\delta\phi\,\mathrm{d}V=\iiint(\nabla^2\phi)\,\delta\phi\,\mathrm{d}V+\iint(\phi_n-n_t)\,\delta\phi\,\mathrm{d}S$$
$$-\iint[\phi_t+\tfrac{1}{2}(\nabla\phi)^2+g\zeta]_{z=\zeta}\,\delta\zeta\,\mathrm{d}x\,\mathrm{d}y. \quad (2.9)$$

Invoking the requirement $\delta J=0$ for variations $\delta\phi$ and $\delta\zeta$ that vanish at the temporal end points of the action integral J but are otherwise arbitrary, we obtain (2.1)–(2.4).

An equivalent Lagrangian, which typically is more convenient for computation, may be derived from (2.6) by transforming the integrals of ϕ_t and $\frac{1}{2}(\nabla\phi)^2$ through the replacement of $\delta\phi$ by ϕ in (2.7) and (2.8) and invoking Gauss's theorem to obtain

$$\iiint gz\,\mathrm{d}V=\frac{1}{2}\iint_{F+W}gz^2\,\mathrm{d}x\,\mathrm{d}y-\tfrac{1}{2}gd^2b[l-\chi(-d,\,t)], \quad (2.10)$$

where F is the free surface and W is the wavemaker. The end result is

$$L\equiv\hat{L}+\partial_t\iiint\phi\,\mathrm{d}V-\tfrac{1}{2}gd^2b[l-\chi(-d,\,t)] \quad (2.11a)$$

$$=\frac{1}{2}\iiint\phi\nabla^2\phi\,\mathrm{d}V+\iint\phi(\tfrac{1}{2}\phi_n-n_t)\,\mathrm{d}S-\tfrac{1}{2}g\iint_{F+W}z^2\,\mathrm{d}x\,\mathrm{d}y. \quad (2.11b)$$

The difference $L-\hat{L}$ makes a null contribution to δJ, by virtue of which $\delta\int L\,\mathrm{d}t=0$ also implies (2.1)–(2.4).† Expressing ϕ_n and n_t in Cartesian coordinates on F and W and assuming that $\phi_n=0$, i.e. (2.3), is satisfied on the remaining boundaries, we obtain

$$L=\frac{1}{2}\int_0^b\mathrm{d}y\left\{\iint\phi\nabla^2\phi\,\mathrm{d}x\,\mathrm{d}z+\int_{x_0}^{l}[\phi(2\zeta_t-\phi_z+\nabla\phi\cdot\nabla\zeta)-g\zeta^2]_{z=\zeta}\,\mathrm{d}x\right.$$
$$\left.+\int_{-d}^{z_0}[\phi(\phi_x-\nabla\chi\cdot\nabla\phi-2\chi_t)-gz^2\chi_z]_{x=\chi}\,\mathrm{d}z\right\}, \quad (2.12)$$

where $x_0(y,\,t)$ and $z_0(y,\,t)$ are the coordinates of the intersection of the wavemaker $(x=\chi)$ and the free surface $(z=\zeta)$.

3. Trial functions

We posit the trial functions

$$\left(\frac{k\omega}{g}\right)\phi=\epsilon\phi_0+\epsilon^{\frac{1}{2}}\phi_1+\epsilon\phi_{11}+O(\epsilon^{\frac{3}{2}}), \quad (3.1a)$$

$$k\zeta=\epsilon\zeta_0+\epsilon^{\frac{1}{2}}\zeta_1+\epsilon\zeta_{11}+O(\epsilon^{\frac{3}{2}}), \quad (3.1b)$$

where the dimensionless variables $(\phi_0,\,\zeta_0)$ represent the first-order (linear) plane-wave solution of (2.1)–(2.4), $(\phi_1,\,\zeta_1)$ represent the first-order cross-wave solution, and $(\phi_{11},\,\zeta_{11})$ represent the second-order interaction of $(\phi_1,\,\zeta_1)$ with itself.

It suffices for the present calculation to know that the first-order plane-wave solution (cf. Havelock 1929) is independent of y and satisfies

$$\phi_{0xx}+\phi_{0zz}=0, \quad (3.2)$$

† It is evident that (2.11) is relevant to other moving-boundary problems – e.g. a floating body.

$$\phi_{0x} = 2\kappa f(z)\cos 2\omega t \quad (x=0), \qquad \int_0^l \phi_{0z}|_{z=0}\,\mathrm{d}x = 2\kappa\int_{-\infty}^0 f(z)\,\mathrm{d}z\cos 2\omega t \quad (3.3\,a,b)$$

within $1+O(\epsilon)$ ((3.3b) follows from continuity; cf. Garrett 1970).

The second-order cross-wave solution of (2.1)–(2.3) for $k=\kappa$ may be inferred from Rayleigh's (1915) second-order solution for two-dimensional (y, z in the present context) standing waves. Matching Rayleigh's result to (1.6), we obtain

$$\phi_1 = \surd 2A_\theta(\theta;\tau)\cos ky\,\mathrm{e}^{kz}, \quad \zeta_1 = \surd 2A(\theta;\tau)\cos ky, \tag{3.4a,b}$$

$$\phi_{11} = -AA_\theta, \quad \zeta_{11} = A^2\cos 2ky, \tag{3.5a,b}$$

where
$$A(\theta,\tau) = p(\tau)\cos\theta + q(\tau)\sin\theta = \mathscr{R}\{(p+\mathrm{i}q)\,\mathrm{e}^{-\mathrm{i}\theta}\} \tag{3.6}$$

and
$$\theta = \omega t, \quad \tau = \epsilon\omega t. \tag{3.7a,b}$$

4. The average Lagrangian

We next substitute the trial functions (3.1a, b) into (2.12) and average the result over θ with τ fixed to obtain the average Lagrangian $\langle L\rangle$ as a functional of p and q. The volume integral vanishes by virtue of ∇^2 (3.1a) $=0$, so that we need consider only the surface integrals. In evaluating these integrals, it is expedient to separate out the contributions of the end points through the approximation

$$\int_{x_0}^l [\;]\,\mathrm{d}x = \int_0^l [\;]\,\mathrm{d}x - x_0[\;]_{x=0} + O(x_0^2) \tag{4.1}$$

and similarly for the integral over the wavemaker.

Considering first the free-surface integral, replacing ζ_t by $\zeta_t+\epsilon\omega\zeta_\tau$ (this is the only term in L in which the τ-derivative is significant in the present approximation), regrouping the terms in the integrand, averaging over θ, and invoking (3.2)–(3.7) and $x_0 = \chi(0,t)\,[1+O(\epsilon^{\frac{1}{2}})]$, we obtain

$$\begin{aligned}
\frac{1}{2}\int_0^b \mathrm{d}y\left\langle \int_{x_0}^l [2\epsilon\omega\phi\zeta_\tau + (\phi\zeta_t - g\zeta^2) + \phi(\zeta_t - \phi_z + \phi_x\zeta_x + \phi_y\zeta_y)]_{z=\zeta}\,\mathrm{d}x\right\rangle \\
= (0) + \tfrac{1}{2}ga^2 l\int_0^b \mathrm{d}y\Big\langle 2\phi_1\zeta_{1\tau} + \epsilon^{-1}(\phi_1\zeta_{1\theta} - \zeta_1^2) + \epsilon^{-1}\phi_1\left(\zeta_{1\theta} - \frac{k}{\kappa}\phi_1\right) \\
+ (\phi_{11}+\phi_1\zeta_1)(2\zeta_{11\theta} - \phi_1\zeta_1) - \zeta_{11}^2 + \frac{1}{kl}\int_0^l (\phi_{0z} - \kappa^{-1}\phi_{0zz})\,\mathrm{d}x\,\phi_1\zeta_1 \\
+ (k\kappa)^{-1}\phi_{1y}(\phi_1\zeta_{11y} + 2\phi_1\zeta_1\zeta_{1y} + \phi_{11}\zeta_{1y}) - (kl)^{-1}(\phi_1\zeta_{1\theta} - \zeta_1^2)f(0)\sin 2\theta\Big\rangle_{z=0} \quad (4.2a) \\
= (0) + \tfrac{1}{2}ga^2 bl\left[\dot{p}q - p\dot{q} + \beta(p^2+q^2) + \tfrac{1}{8}(p^2+q^2)^2 + \left(l^{-1}\int_{-\infty}^0 f\,\mathrm{d}z\right)pq\right], \quad (4.2b)
\end{aligned}$$

where, here and subsequently, error factors of $1+O(\epsilon^{\frac{1}{2}})$ are implicit, (0) stands for terms that depend only on the plane-wave solution (and therefore are independent of p and q), $\dot{p}\equiv \mathrm{d}p/\mathrm{d}\tau$, and

$$\beta = \frac{1}{2\epsilon}\left(1-\frac{k}{\kappa}\right) = \frac{\omega^2-\omega_1^2}{2\epsilon\omega^2} \tag{4.3}$$

is a measure of the proximity to resonance (cf. (1.4)). Small surface tension may be incorporated by multiplying k and ω_1^2 by $1+\hat{T}$; see (1.5).

Turning to the integral over the wavemaker and invoking $z_0 \doteqdot \zeta|_{x=0}$, and the implicit satisfaction of (2.4) by ϕ_0, we obtain

$$\frac{1}{2}\int_0^b \mathrm{d}y \left\langle \int_{-\infty}^{z_0} [\phi(\phi_x - \chi_z\phi_z - 2\chi_t) - gz^2\chi_z]_{x=\chi}\,\mathrm{d}z \right\rangle$$

$$= (0) - \tfrac{1}{2}ga^2\int_0^b \mathrm{d}y \left\langle \int_{-\infty}^{0} [2\phi_{11} f\cos 2\theta + \kappa^{-1}\phi_1^2 f' \sin 2\theta]\,\mathrm{d}z + 2k^{-1}(\phi_1 f\zeta_1)_{z=0}\cos 2\theta \right\rangle \tag{4.4a}$$

$$= (0) + \frac{1}{2}\frac{ga^2b}{\kappa}\left[\int_{-\infty}^{0} (\kappa f + \tfrac{1}{2}f'\,\mathrm{e}^{2\kappa z})\,\mathrm{d}z - f(0)\right]pq. \tag{4.4b}$$

Combining (4.2) and (4.4) in $\langle(2.12)\rangle$ and subtracting out the plane-wave ($p = q = 0$) Lagrangian L_0, we obtain

$$\mathscr{L} \equiv \frac{\langle L - L_0\rangle}{ga^2bl} = \tfrac{1}{2}(\dot{p}q - p\dot{q}) + H(p,q), \tag{4.5a}$$

where
$$H = \tfrac{1}{2}\beta(p^2+q^2) + \beta_* pq + \tfrac{1}{16}(p^2+q^2)^2 \tag{4.5b}$$

and
$$\beta_* = \frac{1}{l}\int_{-\infty}^{0} f(z)\,\mathrm{d}z + (4\kappa l)^{-1}\left[\int_{-\infty}^{0} f'(z)\,\mathrm{e}^{2\kappa z}\,\mathrm{d}z - 2f(0)\right]. \tag{4.6}$$

We remark that, although the implicit error factor in (4.5a) is $1+O(\epsilon^{\frac{1}{2}})$, the corresponding error factor in the subsequent evolution equations is $1+O(\epsilon)$ by virtue of the variational principle that the error in $\delta\int\mathscr{L}\,\mathrm{d}\tau$ is of the order of the square of the error in the trial function.

The parameter β_* is a measure of the energy transfer from the wavemaker to the cross-waves. The first integral in (4.6) is derived equally from the wavemaker and from the integral of $\langle\phi_{0z}\phi_1\zeta_1\rangle$ over the free surface (the corresponding energy is, of course, derived originally from the wavemaker) through the equality (3.3b); the remaining terms are derived from the wavemaker. We may assume $\beta_* > 0$ without loss of generality, since, from (4.6), $\beta_* \to -\beta_*$ is equivalent to $f(z) \to -f(z)$, which, from (1.1), is equivalent to $\omega t \to \omega t + \frac{1}{2}\pi$. Evaluating the integrals for a flap hinged at $z = -d$, for which

$$f(z) = 1 + \frac{z}{d} \quad (-d \leqslant z \leqslant 0), \tag{4.7}$$

we obtain
$$2\kappa l\beta_* = (4\kappa d)^{-1}(2\kappa d - 1)^2. \tag{4.8}$$

The fraction of the total energy transfer that is derived from the free surface (see above) is $\frac{1}{2}[2\kappa d/(2\kappa d-1)]^2$, which exceeds $\frac{1}{2}$ for $\kappa d > \frac{1}{4}(4\pi d > b)$.

5. Evolution equations

Requiring $\int\mathscr{L}\,\mathrm{d}\tau$ to be stationary with respect to independent variations of p and q, we obtain the evolution equations

$$\dot{p} = -\frac{\partial H}{\partial q} = -\beta_* p - [\beta + \tfrac{1}{4}(p^2+q^2)]q \tag{5.1a}$$

and
$$\dot{q} = \frac{\partial H}{\partial p} = \beta_* q + [\beta + \tfrac{1}{4}(p^2+q^2)]p, \tag{5.1b}$$

in which H appears as a Hamiltonian and p and q are canonically conjugate variables. It follows that H is a constant of the motion, by virtue of which the general solution may be reduced to quadrature through the introduction of action-angle variables (which lead to elliptic integrals; cf. Miles 1984*a*).

The fixed points of (5.1), at which $\dot{p} = \dot{q} = 0$ and the wave motion is harmonic, are given by

$$p = q = 0, \tag{5.2a}$$

$$p = -q = \pm[2(\beta_* - \beta)]^{\frac{1}{2}} \quad (\beta < \beta_*) \tag{5.2b}$$

and

$$p = q = \pm[-2(\beta_* + \beta)]^{\frac{1}{2}} \quad (\beta < -\beta_*) \tag{5.2c}$$

The assumption of small disturbances, proportional to $\exp(\lambda\tau)$, with respect to these fixed points yields

$$\lambda^2 = \beta_*^2 - \beta^2, \tag{5.3a}$$

$$\lambda^2 = 4\beta_*(\beta - \beta_*) \quad (\beta < \beta_*) \tag{5.3b}$$

and

$$\lambda^2 = 4\beta_*(-\beta - \beta_*) \quad (\beta < -\beta_*), \tag{5.3c}$$

respectively. It follows that the fixed point at $p = q = 0$ (plane-wave motion without cross-waves) is stable/unstable for $\beta^2 \gtrless \beta_*^2$ and that those of (5.2*b*/*c*) are stable/unstable. Both plane-wave ($p = q = 0$) and cross-wave (with p, q given by (5.2*b*)) motion are stable for $\beta < -\beta_*$, and which is realized depends on the initial conditions.

The criterion $\beta^2 > \beta_*^2$ for stability of the plane-wave solution is equivalent in form to that determined by Garrett (1970), but he obtains

$$\beta_* = (2\kappa l)^{-1}\left[2\kappa\int_{-\infty}^{0} f(z)\,\mathrm{d}z - f(0)\right] \tag{5.4}$$

in place of (4.6) in consequence of his linearization of the boundary condition at the wavemaker. The difference between (4.6) and (5.4) for a flap hinged at $z = -d$ is $(b^2/8\pi^2 ld)$, which is typically small; however, this difference would be larger for a flap hinged closer to the surface.

The details of the phase-plane trajectories implied by (5.1)–(5.3) may be inferred from those for Faraday resonance of surface waves in a cylinder that is subjected to a vertical oscillation. The canonical transformation (which is equivalent to a scale change and a $\frac{1}{4}\pi$ phase shift of the complex amplitude)

$$p = (2\beta_*)^{\frac{1}{2}}(P - Q), \quad q = (2\beta_*)^{\frac{1}{2}}(P + Q), \quad \tau = \beta_*^{-1}T, \tag{5.5a, b, c}$$

carries (4.6) over to

$$\mathscr{L} = 4\beta_*^2[\tfrac{1}{2}(P_T Q - PQ_T) + \hat{H}(P, Q)] \tag{5.6a}$$

and

$$\hat{H} = \frac{1}{2}\left(\frac{\beta}{\beta_*}\right)(P^2 + Q^2) + \tfrac{1}{2}(P^2 - Q^2) + \tfrac{1}{4}(P^2 + Q^2)^2, \tag{5.6b}$$

which is equivalent to the corresponding system for the Faraday-resonance problem (Miles 1984*a*) with β replaced by β/β_* therein. Invoking this equivalence, we find that the fixed points may be classified as follows (see figure 2 in Miles 1984*a*):

(*a*) $\beta > \beta_*$, centre at $p = q = 0$;

(*b*) $-\beta_* < \beta < \beta_*$, saddle point at $p = q = 0$ and two centres at $p = -q = \pm[2(\beta_* - \beta)]^{\frac{1}{2}}$,

(*c*) $\beta < -\beta_*$, three centres at $p = q = 0$ and $p = -q = \pm[2(\beta_* - \beta)]^{\frac{1}{2}}$ and two saddle points at $p = q = \pm[-2(\beta_* + \beta)]^{\frac{1}{2}}$.

Weak dissipation (see below) will cause the phase-plane trajectories to spiral into the centres, which correspond to stable plane-wave motion in (*a*) and cross-wavecontaminated motion in (*b*). The trajectories about $p = q = 0$ in (*c*) correspond to $-(\beta+\beta_*)^2 < H < 0$, with $H>0$ for the trajectories about the remaining centres. It follows, since H is determined by the initial conditions, that $H \lessgtr 0$ implies plane-wave/cross-wave motion if $\beta < -\beta_*$.

The incorporation of weak, linear damping in the dynamical formulation leads to the introduction of $\alpha(p,q)$ on the left-hand sides of (5.1*a*, *b*), where (cf. Miles 1984*a*)

$$\alpha \equiv \frac{\delta}{\epsilon}, \tag{5.7}$$

and δ is the ratio of actual to critical damping for the pure cross-wave (which, without excitation, would decay like $\exp(\delta\omega_1 t)$). We also introduce

$$\gamma \equiv (\beta_*^2 - \alpha^2)^{\frac{1}{2}}. \tag{5.8}$$

The fixed points then may be classified as follows if $\alpha < \beta_*$:

(*a*) $\beta > \gamma$, sink at $p = q = 0$;

(*b*) $-\gamma < \beta < \gamma$, saddle point at $p = q = 0$ and two sinks at

$$p + \mathrm{i}q = \pm 2 \exp\left[\mathrm{i}\left(\frac{3\pi}{4} - \phi\right)\right](\gamma - \beta)^{\frac{1}{2}}, \tag{5.9}$$

where

$$\cos 2\phi \equiv \frac{\gamma}{\beta_*} = \left(1 - \frac{\alpha^2}{\beta_*^2}\right)^{\frac{1}{2}}; \tag{5.10}$$

(*c*) $\beta < -\gamma$, three sinks at $p = q = 0$ and at (5.9), and two saddle points at

$$p + \mathrm{i}q = \pm 2 \exp\left[\mathrm{i}(\tfrac{1}{4}\pi + \phi)\right](-\gamma - \beta)^{\frac{1}{2}}. \tag{5.11}$$

The only fixed point for $\alpha > \beta_*$ is a sink at $p = q = 0$ – i.e. cross-waves will decay if

$$\delta > \beta_* \epsilon = \frac{a}{4l}\left[f(0) + 2\kappa \int_{-\infty}^{0} f(z)\,(2 - \mathrm{e}^{2\kappa z})\,\mathrm{d}z\right]. \tag{5.12}$$

Conversely, cross-waves may occur in the appropriate ranges of β if a exceeds the threshold amplitude obtained by letting $\delta = \beta_* \epsilon$.

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